5

Physicists' Approach to Studying Socio-Economic Inequalities

Can Humans be Modelled as Atoms?

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Introduction

Physicists have been always keen on exploring domains outside of physics, like biology, geology, astronomy, sociology, economics, etc., often giving birth to very successful interdisciplinary subjects like biophysics, astrophysics, geophysics, sociophysics, econophysics, and so on (Chakraborti, Raina and Sharma 2016). The last two interdisciplinary fields: sociophysics (Sen and Chakrabarti 2013; Chakrabarti, Chakraborti and Chatterjee 2006) and econophysics (Sinha, Chatterjee, Chakraborti and Chakrabarti 2010; Slanina 2013), have been only recent additions to the long list. However, the physicists' interest in the social sciences (economics and sociology) is quite old, and they have been trying to approach economic and social problems using their experience of modelling physical systems and analyzing data. The physicists' ability to deal with complex dynamical systems have often inspired the central ideas and foundations of the modern axiomatic economics. No wonder, the first Nobel Prize winner in economics was Jan Tinbergen, a physicist by training (having completed his PhD from the University of Leiden in 1929 on 'Minimization problems in Physics and Economics').[1] The book of Paul Samuelson, the second Nobel Laureate in economics, entitled *Foundations of Economic Analysis* (1947), which is considered as his magnum opus, derived from his doctoral dissertation at Harvard University, makes use of the classical thermodynamic methods of Willard

Gibbs, the American physicist and one of the founders of the area of statistical physics. On the other hand, the economic concept of the 'invisible hand' attributable to Adam Smith, cited as the father of modern economics and best known for his two classic works—*The Theory of Moral Sentiments* (1759), and *An Inquiry into the Nature and Causes of the Wealth of Nations* (1776)—can be understood as an attempt to describe the influence of the market as a spontaneous order on people's actions and self-organization, which has influenced many models of physical systems. Many such cross-fertilization of ideas and concepts have been taking place for a long time, eventually leading to intense activities in the field and the coinage of the word econophysics.

In the field of statistical physics(Mandl 2002; Haar 1995; Sethna 2006), one often encounters a system of many interacting dynamical units exhibiting a 'collective behavior', which simply depends on a few basic (dynamical) properties of the individual constituents and the embedding dimension of the system. Since it is independent of other details, it displays a sort of 'universality'. Often, socio-economic data also exhibit enough empirical evidences in support of such 'universalities', which prompts the physicists to propose simple, minimalistic models to understand them using the methods of statistical physics.

In sociology and economics, some of the major issues of concern have been inequalities in different forms: income, wealth, etc. It has been argued by certain social scientists that a society or country performs better when the resources are distributed more equitably, or there exists less inequalities between the haves and the have-nots (Wilkinson and Pickett 2009). Most economists agree that fairness or equal chances of being involved in the economic activities promote growth (Rodrik 2003). However, it would be impossible to find any society or country where, e.g. income or wealth, is equally distributed among its people. The distribution of wealth, income, and consumption has never been uniform, and economists (and very recently physicists) have tried for years to understand the reasons for such inequalities. For a very long time, scholars have been working on the statistical descriptions and mechanisms leading to such in equalities (see, e.g. Sen 1992; Chakrabarti, Chakraborti, Chakravarty and Chatterjee 2013; Deaton 1992). We briefly mention a few of the empirical observations in the first section of the essay. Based on these observations, many questions have been formulated by different scholars. In the following sections, we address, from the perspective of the physicists, the questions below:

(i) How are income, wealth, and consumption distributed and what are the statistical forms of their distributions?
(ii) Are there any robust or 'universal' features of the statistical forms and how can they be modelled/reproduced in a mathematical/computational framework?

Empirical Distributions of Income and Expenditure

Following several studies spanning more than a century, a few established regularities in income and wealth distributions have been observed. The most popular regularity was proposed by the Italian sociologist and economist Vilfredo Pareto, who made extensive studies at the end of the nineteenth century and found that wealth distribution in Europe follows a power law for the very rich (Pareto 1897). Later this came to be known as the Pareto law. Subsequent studies revealed that the distributions of income and wealth possess other fairly robust features, like the bulk of both the income and wealth distributions seem to reasonably fit both the log-normal and the gamma distributions, sometimes also known as Gibrat's law (Gibrat 1931).

Physicists use the gamma distribution for fitting the probability density or the Boltzmann-Gibbs/exponential distribution for the corresponding cumulative distribution. However, the tail of the distribution fits well to a power law (as first observed by Pareto), with the exponent, known as the Pareto exponent, usually ranging between 1 and 3 (Chakrabarti, Chakraborti, Chakravarty and Chatterjee 2013). For India too, the wealthiest have been found to have their assets distributed along a power law tail (Sinha 2006). The shape of the typical wealth distribution is thus, with Gibbs/Gamma behaviour at lower and intermediate values of wealth $w<w_c$, and a Pareto (power law) tail at the larger values, $w \geq w_c$, where w_c is a crossover value that depends on the numerical fitting of the data:

$$
\begin{aligned}
P(w) &\sim w^n exp(-w/T), && \text{for } \; w < w_c \\
&\sim w^{-\alpha-1}, && \text{for } \; w \geq w_c \qquad (1)
\end{aligned}
$$

Pareto Law. In 1897, Pareto made extensive studies in Europe and found that wealth distribution follows a power law tail for richer sections of society. For about 90–95 per cent of the population, the distribution matches a Gibbs or Gamma (black curve), while the income for the top 5–10 per cent of the population decays much more slowly, following a power law (bold line).

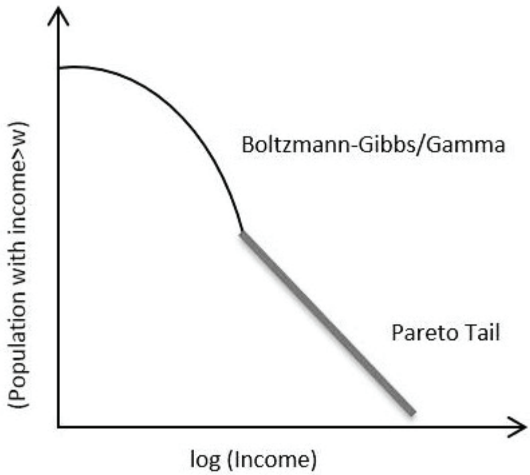

FIGURE 5.1: Plot for the cumulative income distribution

Source: Prepared by the authors.
Note: The population fraction having an income greater than a value w (plotted against income) is shown on a double logarithmic scale. The exponent of the Pareto tail, known as the Pareto exponent, is given by the slope of the bold line in the double-logarithmic scale, and is found to be between 1 and 3 (see Chakrabarti, Chakraborti, Chakravarty and Chatterjee 2013; Pareto 1897).

where $P(w)$, the equilibrium distribution of wealth, is defined as follows: $P(w)dw$ is the probability that in the steady state of the system, a randomly chosen agent will be found to possess wealth between w and $w + dw$. The exponent α is known as the Pareto exponent, T is the average wealth (analogous to the temperature in a gas) of the economic system, and n is a numerical constant. Detailed empirical results in support of the given statistical form for both income and wealth in different countries, economic societies, and over different periods of time, can be found in several research articles, monographs, and books that are mentioned in the list of references. For an illustration, Fig. 5.2 shows the cumulative probability distribution of the net wealth, composed of assets (including cash, stocks, property, and household goods) and liabilities (including mortgages and other debts) in the United Kingdom for the year 1996.

Although income and wealth distribution data are mainly used to quantify economic inequality for individuals or family/households, the distribution of consumer expenditure also reflects certain aspects of disparity in the society. In a recent study, Chakrabarti et al. (2018) analyzed the unit-level expenditure on

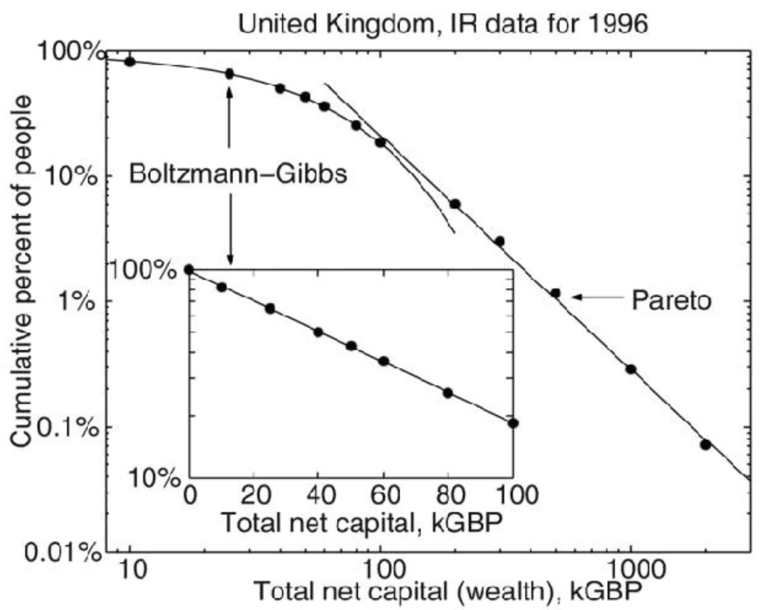

FIGURE 5.2: Cumulative probability distribution of the net wealth
Source: Dragulescu and Yakovenko 2001.
Note: The wealth composed of assets (including cash, stocks, property, and household goods) and liabilities (including mortgages and other debts) in the United Kingdom is shown on double-logarithmic (main panel) and log-linear (inset) scales. Points represent the data from the Inland Revenue, and solid lines are fits to the Boltzmann-Gibbs (exponential) and Pareto (power) distributions.

consumption across multiple countries and multiple years, and showed that certain invariant features of consumption distribution could be extracted. Specifically, it was shown that the bulk of the distribution follows a log-normal, followed by a power law tail. As shown in Fig. 5.3, the distributions coincide with each other under normalization by mean expenditure and log scaling, even though the data was sampled across multiple dimensions including time, social structure, and locations across the globe. This observation seems to indicate that the dispersion in consumption expenditure across various social and economic groups are significantly similar ('universal'), subject to suitable scaling and normalization. In another article, Chatterjee et al. (2016) studied the distributional features and inequality of consumption expenditure, specifically across India for different states, castes, religions, and urban-rural divide. Once again, they found that even though the aggregate measures of inequality are fairly diversified across Indian states, the consumption distributions show near identical statistics after proper normalization. This feature was again seen to be robust with respect to variations in sociological and economic factors. They also showed that statewise inequality seems to be positively correlated with growth, which is in agreement with the traditional idea of first part of the Kuznets curve (Kuznets 1955).

Having discussed briefly the first question mentioned in the introduction, we now turn our attention to the modelling of the different robust features of empirical distributions, by using a mathematical/computational framework that is inspired by some simple models of statistical physics of ideal gas.

Kinetic Exchange Models

The simple yet powerful framework of kinetic theory of ideal gases, first proposed in 1738 by Bernoulli, led eventually to the successful development of statistical physics towards the end of the nineteenth century (Mandl 2002; Haar 1995; Sethna 2006). The aim of statistical physics is to study the physical properties of macroscopic systems consisting of a large number of constituent particles. In such large systems, the number of particles is of the order of Avogadro number, and it is extremely difficult to have complete microscopic description of such a system.

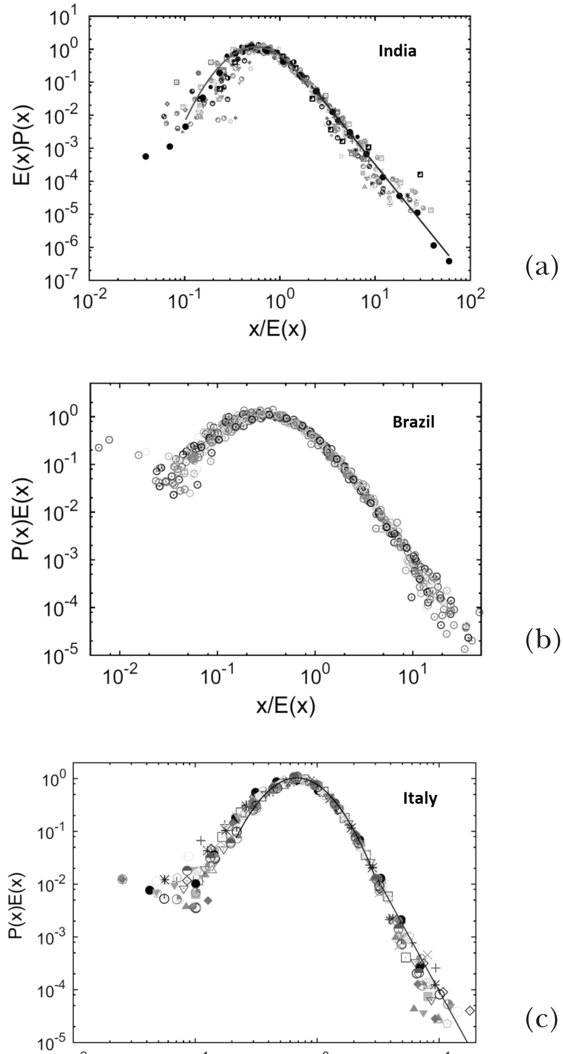

FIGURE 5.3: Plots for consumption expenditure data: (a) India, (b) Brazil, and (c) Italy

Source: Chakrabarti et al. 2018.

Note: (a) For Indian states, data has been normalized for all states and fitted with a log-normal distribution and a power law at the right tail during the period 2011–12. (b) For Brazilian states, data has been normalized with respect to the respective mean expenditure across states during the period 2008–9. (c) For all states of Italy, performed during the period 1980–2012.

The basic concept of the kinetic exchange model is taken from the 'Kinetic theory of gases', which describes a gas as a large number of microscopic particles (atoms and molecules), all of which are in constant, random motion. The rapidly moving 'point-like' particles constantly collide with each other or with the walls of the container and exchange kinetic energy. We now describe in details how simple simulations can be used to demonstrate the results of the kinetic theory of 'ideal gases', which can be adapted to model simple, closed economic systems for studying income/wealth distribution.

KINETIC ENERGY EXCHANGE MODEL ('IDEAL GAS')

Kinetic exchange models are stochastic models, which are interpreted in terms of energy exchanges in gas molecules. The kinetic exchange model describes the dynamics at a microscopic level, based on pairwise molecular collisions. Boltzmann wrote that 'molecules are like so many individuals, having the most various states of motion' (Boltzmann 1872). Thus, for two particles i and j with energies $w_i(t)$ and $w_j(t)$ at time t, the general dynamics can be described by the mathematical equations:

$$w_i(t+1) = w_i(t) + \Delta w;$$
$$w_j(t+1) = w_j(t) - \Delta w; \tag{2}$$

where time t changes by one unit after each collision. A typical energy exchange process is shown schematically in Fig. 5.4.

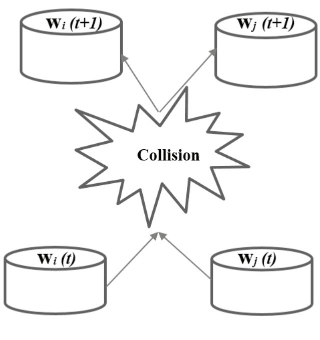

FIGURE 5.4: Example of kinetic energy exchange

Note: Two particles i and j taking part in energy exchange ('collision') process. Particles i and j have energies $w_i(t)$ and $w_j(t)$ at time t. After collision, their energies become $w_i(t+1)$ and $w_j(t+1)$, respectively.

The Boltzmann-Gibbs distribution,[5] a fundamental law of equilibrium statistical mechanics, states that the probability P(w) of finding a physical system or subsystem in a state with energy w is given by the exponential function:

$$P(w) = Ce^{-w/T}, \tag{3}$$

where T is the temperature (average kinetic energy) of the system, C is a constant, and the conserved quantity is the total energy of the system.

SIMULATION OF THE KINETIC ENERGY EXCHANGE MODEL

Assume that N interacting units i, with $i = 1, 2, \ldots, N$, are molecules of a gas with no interaction (potential) energy, and the variables of w_i represent their kinetic energies, such that $w_i \geq 0$. The time evolution of the system proceeds by a discrete stochastic dynamics (Patriarca and Chakraborti 2013). A series of updates of the kinetic energies $w_i(t)$ are made at discrete times $t = 0, 1, \ldots$. Each update takes into account the effect of a collision between two molecules (as shown in the schematic diagram, Fig. 5.4). The time step, which can be set to $\Delta t = 1$ without loss of generality, represents the average time interval between two consecutive molecular collisions; i.e. on average, after each time step Δt, two molecules i and j undergo a 'scattering' process, and an update of their kinetic energies w_i and w_j is made. The evolution of the system is accomplished by the following steps at each time t:

(i) Randomly choose a pair of molecules i and j $(i \neq j)$ and $1 \leq I, j \leq N$, with kinetic energies w_i and w_j respectively; they represent the molecules undergoing a collision.

(ii) Perform the energy exchange between molecules i and j by updating their kinetic energies,

$$w_i(t+1) = r_t\,[w_i(t) + w_j(t)],$$
$$w_j(t+1) = (1 - r_t)[w_i(t) + w_j(t)], \tag{4}$$

where r_t is a stochastic variable drawn as a uniform random number between 0 and 1, at time t. The total kinetic energy is conserved during an interaction.

(iii) Increment the time step, and go to first step (see the MATLAB code given in the appendix to the chapter).

For a large number of molecules ($N \to \infty$) and a sufficient number of time steps ($t \to \infty$), the system reaches an equilibrium (or steady-state) in distribution (Goswami and Chakraborti 2015). The equilibrium distribution turns out to be the Boltzmann-Gibbs (exponential) distribution, as shown in Fig. 5.5, which can be derived analytically in several ways—probabilistic calculations (Dragulescu and Yakovenko 2000), Master equation (Chatterjee, Chakrabarti and Stinchcombe 2005), variational principle of maximum entropy (Chakraborti and Patriarca 2009), etc. Interestingly, the most probable value of the exponential equilibrium distribution is zero (or very little energy).

Kinetic Wealth Exchange Models

As mentioned in the introduction, understanding the distributions of income and wealth in an economy has been a classic problem in economics for more than a hundred years (Chakrabarti, Chakraborti, Chakravarty and Chatterjee 2013). Inspired from the kinetic theory of gases (mentioned in the last section), the kinetic wealth exchange models (KWEMs) were proposed, which tried to explain the robust and universal features of income/wealth distributions. These form a class of simple multi-agent models, where the actions and interactions of autonomous agents (representing individuals, organizations, societies, etc.) could be used to understand the behaviour of the system as a whole (Yakovenko and Rosser Jr 2009; Chatterjee and Chakrabarti 2007; Chatterjee 2010; Chakraborti 2002; Hayes 2002; Lallouache, Jedidi and Chakraborti 2010; Matthes and Toscani 2008a, 2008b; Comincioli, Croce and Toscani 2009).

KWEMs owe their popularity to the fact that they can capture many of the robust features of realistic wealth distributions using a minimal set of exchange rules (Patriarca and Chakraborti 2013; Chakraborti 2002; Patriarca, Chakraborti and Kaski 2004). In KWEMs, the closed economy or society is described in terms of a simple model, in which agents randomly meet and exchange a part of their wealth (Patriarca and Chakraborti 2013), similar to particle assemblies (e.g. a gas) in which, from time to time, a pair of particles collide and exchange energy, as given in Eq. 2. The core of the model dynamics are the simple linear relations in Eq. 2, and the difficulty is actually in generalizing and adapting the models, and solving analytically the equations. In fact, the exchange of wealth between two agents parallels the exchange of energy between colliding particles to the point that the kinetic theory of a gas in D dimensions can suggest the expressions for the equilibrium distributions.

MODEL WITH NO SAVING

The first model of this type was introduced by J. Angle in the context of social science (see Angle 1986, 2006), some years earlier than in physics or economics. In the 1960s, Mandelbrot had suggested the possibility '… to consider the exchanges of money which occur in economic interaction as analogous to the exchanges of energy which occur in physical shocks between gas molecules

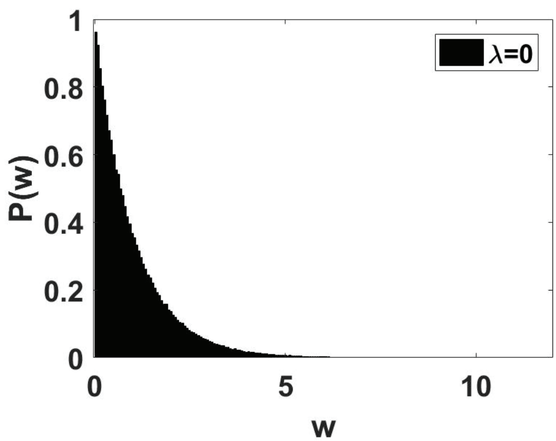

FIGURE 5.5: Equilibrium energy distribution of the kinetic energy exchange model

Note: The results are obtained using the MATLAB code (provided in the appendix), with parameters: $N = 200$ particles and $T = 50000$ time steps, and averaged over an ensemble of $k = 2000$ runs. The distribution corresponds to the exponential Boltzmann-Gibbs distribution (Eq. 3). The convergence to equilibrium is fast.

…' (Mandelbrot 1960: 83), but it was not until the works of E. Bennati (1988a, 1988b, 1993) that such an analogy between statistical mechanics and the economics of wealth exchange was realized in terms of a quantitative Monte Carlo model, for which the corresponding numerical simulations demonstrated that the Boltzmann-Gibbs distribution was the equilibrium wealth distribution. Later, many physicists independently discovered such results by Monte Carlo simulations (Dragulescu and Yakovenko 2000; Ispolatov et al. 1998). However, the introduction of 'saving propensity' (Chakraborti and Chakrabarti 2000), as will be described next, brought forth the gamma-like feature (Patriarca, Chakraborti and Kaski 2004; Chakraborti and Patriarca 2008) of the distribution $P(w)$ and such a kinetic exchange model with uniform saving propensity for all agents was subsequently shown to be equivalent to a commodity clearing market, where each agent maximizes his/her own utility (Chakrabarti and Chakrabarti 2010).

MODEL WITH UNIFORM SAVING

The concept of saving propensity was considered in this framework, first by Chakraborti and Chakrabarti (2000). In this model, the agents save a fixed fraction $0 \leq \lambda \leq 1$ of their wealth, when interacting with another agent. Thus, two agents with initial wealth $w_i(t)$ and $w_j(t)$ at time t interact such that they end up with wealth $w_i(t + 1)$ and $w_j(t + 1)$ given by:

$$w_i(t + 1) = \lambda\, w_i(t) + r_t\, [(1- \lambda)(w_i(t) + w_j(t))],$$
$$w_j(t + 1) = \lambda\, w_j(t) + (1-r_t)[(1- \lambda)(w_i(t) + w_j(t))], \qquad (5)$$

where r_t is a stochastic fraction between 0 and 1, drawn from an uniform random distribution at time t. If $\lambda = 0$, equivalent to Bennati model (Bennati 1988a, 1988b, 1993), then the most probable wealth per agent is zero, and the market is 'non-interacting'. The market dynamics freezes (no interactions occur) when $\lambda = 1$. For the uniform saving propensity λ in between the two limits, ($0 < \lambda < 1$), the steady state distribution $P(w)$ of money is a gamma-like distribution (Patriarca, Chakraborti and Kaski 2004; Chakraborti and Patriarca 2008; Lallouache, Jedidi and Chakraborti 2010) with exponential decaying on both sides and the most-probable money per agent shifting away from 0 (for λ = 0) to the average wealth of the system, *W/N*, as $\lambda \to 1$ (see Fig. 5.6). Note that there is no closed-form analytical solution of this exchange dynamics for the model with $0 < \lambda < 1$ and the gamma distribution is the one which fits closest/best the simulation results (the first three moments matching exactly and the fourth moment differing [Lallouache, Jedidi and Chakraborti 2010]). Here, the 'self-organizing' feature of the market, simply induced by the 'self-interest' of saving by each agent without any global perspective, is quite significant, as the fraction of poor people (with very little or no money) decrease with saving propensity λ, and most people end up with some finite fraction of the average money in the market. The fact that savings can reduce inequality has been studied from a data science perspective in an article by Sharma, Das and Chakraborti (2018). They also studied the empirical data of Gini indices and gross domestic savings (GDS) for several countries, and looked at the co-evolution of the countries in the inequality or savings spaces. Further, they sought an empirical linkage between the income inequality and savings, mainly for relatively small or closed economies, using linear regression model.

Note also that $\lambda \to 1$ corresponds to the case where the economy is ideally 'socialistic' (inequality of wealth is almost zero), and this is achieved just with the self-interest of the people towards saving. Although this fixed saving propensity does not give yet the Pareto-like power law distribution, the Markovian nature of the scattering or trading processes is effectively lost. Indirectly through λ, the agents get to know (start 'interacting' with) each other and the system cooperatively self-organizes towards a non-zero most-probable distribution (see Fig. 5.6). Thus, for $0 < \lambda < 1$, the market is effectively 'interacting' (Goswami and Chakraborti 2015). The relaxation time to reach the steady state distribution is a complicated function of the saving propensity λ and the system size *N* (Patriarca et al. 2007).

Most interestingly for the physicists, the KWEMs in the regime $0 < \lambda < 1$ seem to reproduce the energy dynamics of a *D*-dimensional system, with the additional remarkable feature that the corresponding dimension *D* can assume any real value (Chakraborti and Patriarca 2009; Goswami and Chakraborti 2015; Patriarca, Chakraborti and Kaski 2004), and also establishes interesting links with a generalized *D*-dimensional kinetic theory with real spatial

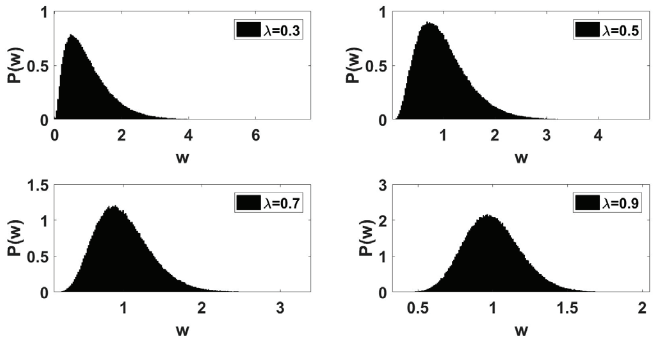

FIGURE 1.6: Steady state wealth distribution with saving propensity λ
Note: Numerical simulations of the model defined by Eq. 5 are best fitted by the gamma distribution (see Patriarca, Chakraborti and Kaski 2004; Lallouache, Jedidi and Chakraborti 2010). The results are obtained using the MATLAB code in the appendix, with parameters: N = 200 particles and T = 50000 time steps, and averaged over an ensemble of k = 2000 runs. The four panels are for λ = 0.3, 0.5, 0.7, and 0.9 respectively. A direct inference that may be drawn from inspection of the figures is that as λ increases, the inequality decreases, which has been studied extensively in Sharma, Das and Chakraborti (2018).

dimension D (Patriarca et al. 2015; Patriarca et al. 2017). Numerical simulations suggest that at equilibrium, the system has a gamma distribution of order n, coinciding with the Boltzmann-Gibbs energy distribution for a D-dimensional gas with $D = 2n$. The relation between the dimension D (or the order $n = D/2$) is: $D = 2(1 + 2\lambda)/(1 - \lambda)$. For $\lambda = 0$, the purely exponential shape is regained.

Model with Distributed Savings

In a real society or economy, the interest of saving varies from person to person, which implies that λ may be a heterogeneous parameter. To mimic this situation, the saving factor λ may be assumed to be widely distributed within the population (Chatterjee, Chakrabarti and Manna 2004, 2003).

As before, starting with an arbitrary initial (uniform or random) distribution of wealth among the agents, the market evolves with trading. At each time, two agents are randomly selected and the wealth exchange among them occurs, following the earlier-mentioned scheme. One checks for the steady state, by looking at the stability of the money distribution in successive Monte Carlo steps t (one Monte Carlo time step is defined as N pairwise exchanges). Eventually, after a typical relaxation time, the wealth distribution becomes stationary. This relaxation time is dependent on system size N and the distribution of λ (e.g. 10^6 for $N = 1000$ and uniformly distributed λ). After this, one averages the money distribution over 10^3 time steps. Finally, one takes the configurational average over 10^5 realizations of the λ distribution to get the money distribution $P(w)$. Interestingly, this non-ergodic model has a power law decay similar to the decay of the Pareto law (see Eq. 1) with $\alpha \approx 1$. One may note, for finite size N of the market, the distribution has a narrow initial growth up to a most-probable value, after which it decays as a power law tail for several decades, and then there is a finite cut-off. This Pareto law ($\alpha \approx 1$) covers almost the entire range in wealth w of the distribution $P(w)$ in the limit $N \rightarrow \infty$. This result can be derived analytically too (Mohanty 2006, Chakraborti and Patriarca 2009).

Discussions

There have been several works and extensions done on these simple KWEMs, and lot of interesting features were extracted (Chakraborti 2002). Recently, a 'bi-directional exchange model' was introduced for mimicking more realistically a wealth exchange (Heinsalu and Patriarca 2014, 2015):

$$w_i(t+1) = r_t\, w_i(t) + q_t\, w_j(t),$$
$$w_j(t+1) = (1-r_t)\, w_i(t) + (1-q_t)\, w_j(t), \tag{6}$$

where r_t and q_t are two independent random numbers in the range of (0, 1) and the sum of the variables before and after the collision is conserved, $w_i(t+1) + w_j(t+1) = w_i(t) + w_j(t)$. In the same paper (Heinsalu and Patriarca 2014), a generalized microscopic version of the model was also introduced, in which, instead of saving propensity parameters, a few parameters regulated probabilistically the microscopic negotiation dynamics between the two agents. For this model, the numerical fitting of the equilibrium distributions suggested an effective dimension D, which is just half of the dimension of the saving propensity model of Chakraborti and Chakrabarti (2000), i.e. $D = (1 + 2\lambda)/(1 - \lambda)$. This result, as well as the analogous ones based on the numerical fittings of the results of other versions of KWEMs, had remained more of a conjecture for some years (Lallouache, Jedidi and Chakraborti 2010; Patriarca, Chakraborti and Kaski 2004; Repetowicz, Hutzler and Richmond 2005). However, this bi-directional exchange model, being more tractable analytically, Katriel (2015) recently managed to show with the help of a Boltzmann equation approach that the relation between the dimension D and the saving parameter λ is exact, thus confirming the deep link between KWEMs and statistical physics. The study of this physics related aspect of the models has now re-entered a challenging active phase in which the theoretical picture of the relaxation and equilibrium in KWEMs is under investigation.

The kinetic exchange framework can suitably be adapted into other areas in economics as well, e.g. the study of the firm size distributions. The size of a firm is measured by the strength of its workers. A firm grows when one worker joins it after leaving another firm. The rate at which a firm gains or loses workers is called the 'turnover rate' in the literature of economics. Thus, there is a redistribution of workers, and the corresponding dynamics can be studied using these exchange models. In the models of firm dynamics, one assumes that:

(i) Any formal unemployment is avoided in the model. Thus one does not have to keep track of the mass of workers who are moving in and out of the employed workers pool.
(ii) The number of workers is treated as a continuous variable.
(iii) The size of a firm is just the number of workers working in the firm.

In firm dynamics models, one makes an analogy with the previous subsections that firms are agents and the number of workers in the firm is its wealth. Assuming no migration, birth, and death of workers, the economy thus remains conserved. As the 'turnover rate' dictates both the inflow and outflow of workers, we need another parameter to describe only the outflow. That parameter may be termed as 'retention rate', which describes the fraction of workers who decide to stay back in their firm. This is identical to saving propensity in wealth exchange models, as discussed earlier. Some interesting results using this framework have been produced by Chakrabarti (2012, 2013).

Emergence of consensus is another important issue in sociophysics problems, where the people interact to select an option among different options of a subject, like vote, language, culture, opinion, etc. (Castellano, Fortunato and Loreto 2009; Galam 2012; Stauffer 2013; Sen and Chakrabarti 2013). When each person chooses an option, often a state of consensus is reached. In the formation of opinions, consensus is analogous to an 'ordered phase' in statistical physics, where most of the people have a particular opinion. Several models have been proposed to mimic the dynamics of spreading opinion, and the opinions are usually modelled as discrete or continuous variables and are subject to either spontaneous changes or changes due to binary interactions, global feedback, and external factors (see Castellano, Fortunato and Loreto [2009] for a general review). Lallouache, Chakraborti and Chakrabarti (2010) and Lallouache et al. (2010) proposed a minimal, multi-agent model for the collective dynamics of opinion-formation in the society, by modifying kinetic exchange dynamics studied in the context of wealth distribution in a society. The model presented an intriguing, spontaneous, symmetry-breaking transition to polarized opinion state starting from non-polarized opinion state (Lallouache et al. 2010), and many other features of interest to the statistical physicists studying phase transitions.

The most interesting aspect of KWEMs is related to their heterogeneous generalizations. In fact, KWEMs owe their popularity to the fact that they can predict realistic wealth distributions using a minimal set of exchange rules (Patriarca and Chakraborti 2013; Chakraborti 2002) but this cannot be achieved in the framework of homogeneous versions of the models, that (as mentioned earlier) usually lead to the exponential or gamma-like equilibrium wealth distributions. Instead, the simple addition of a suitable level of heterogeneity, either in the saving propensities of the agents of the KWEMs of Chakraborti and Chakrabarti (2000), or in the negotiation parameters of the model in Heinsalu and Patriarca (2014) directly leads to the Pareto power law. In other words, KWEMs suggest that heterogeneity is a key factor in producing the power law observed in the wealth distributions (Patriarca et al. 2005; Patriarca, Heinsalu and Chakraborti 2010; Patriarca et al. 2015; Patriarca et al. 2017), a fact related to the general interest toward the effects of diversity in 'Complex Systems' (Patriarca et al. 2015). Also, the dynamics of kinetic exchange models are often criticized for being based on an approach that is far from an actual economic or sociological foundation. However, it has been recently shown, for example, that such an economical dynamics of wealth exchange can also be derived from microeconomic theory (Chakrabarti and Chakrabarti 2010, 2009). Although standard economics theory assumes that the activities of individual agents are driven solely by the utility maximization principle, the alternative picture that was presented is that the agents can also be viewed as particles exchanging 'wealth', instead of energy, and participating in wealth (energy) conserving two-body scattering, as in entropy maximization based on the kinetic theory of gases. This qualitative analogy between the two maximization principles has thus been firmly established only recently.

Final Remarks

Human beings are much more complex than particles! The diverse types of interactions among heterogeneous human beings make society even more complex. However, in a certain idealized and simple closed economy (as mentioned in this essay), we may ignore many complexities or 'degrees of freedom' of the system, and model the system simply as an assembly of atoms or gas particles, in order to reproduce some of the statistical features of the empirical distributions. Definitely, the real economy or society is much more complex, but this is perhaps one baby step of the econophysicists towards modelling the reality!

In this essay, we could give an exposure to only a few models in one particular area. However, the field of econophysics has had many contributions from physicists, economists, mathematicians, financial engineers, and others in the last two decades. Consequently, important directions and new areas in econophysics have emerged in the last two decades, and one could get further information from the following:

- Empirical characterization, analyses, and modelling of financial markets and limit order books (Chakraborti et al. 2011a, 2011b; Abergel et al. 2011; Abergel et al. 2016).

- Network models and characterization of market correlations among different stocks/sectors (Abergel et al. 2012; Sharma et al. 2017; Sharma, Chakrabarti and Chakraborti 2018; Chakraborti et al. 2018).
- Determination of the income or wealth distribution in societies, and the development of statistical physics models (Chatterjee, Yarlagadda and Chakrabarti 2005; Chakrabarti, Chakraborti, Chakravarty and Chatterjee 2013).
- Development of behavioural models, and analyses of market bubbles and crashes (Abergel et al. 2013, 2015; Pharasi et al. 2018).
- Learning in multi-agent game models and the development of Minority Game models (Chakraborti et al. 2015a, 2015b; Sharma et al. 2018).

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Notes

1. See https://en.wikipedia.org/wiki/Jan_Tinbergen, accessed on 13 March 2016.

2. See https://en.wikipedia.org/wiki/Josiah_Willard_Gibbs, accessed on 13 March 2016.
3. See https://en.wikipedia.org/wiki/Adam_Smith, accessed on 13 March 2016.
4. Econophysics, a new interdisciplinary research field applying methods of statistical physics to problems in economics and finance, was introduced by the theoretical physicist H. Eugene Stanley in an international conference on statistical physics at Kolkata (India) in 1995.
5. See https://en.wikipedia.org/wiki/Boltzmann_distribution, accessed on 13 March 2016.

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Appendix

```
% Kinetic Exchange Model
% Model With Saving Propensity

%-Initialization-
clc;
N= 200; %       number of agents
T= 50000; %     number of time steps
K= 2000; %      number of times experiment repeated for taking ensemble average
lambda = 0.0; % saving propensity lambda between 0 and 0.9999
w=ones(N,1); %  initial distribution wealth
y= [ ];      %  initializing an empty matrix to store final distributions

%-Running the model-
for n=1:K       % running model K times
    for t=1:T   % running model for T timesteps
        i= floor(rand()*N)+1;   % choosing agent at i-th position randomly
        j= floor(rand()*N)+1;   % choosing agent at j-th position randomly
        if (i~=j)               % avoid cases where the selected individuals are same
            epsilon= rand();    % stochastic variable
            w_total=w(i)+w(j);
            w(i)=lambda*w(i)+epsilon*(1-lambda)*w_total; %      updated wealth of i
            w(j)=lambda*w(j)+(1-epsilon)*(1-lambda)*w_total;%   updated wealth of j
        end
    end
    y= [y w]; % storing final distributions column-wise in matrix y
end

%-Normalization-
[f,y1]=hist(y,200); % f = matrix of bin frequencies for each column in y,
                    % and y1 = the values corresponding to bins
dy=diff(y1(1:2));   % width of the bins
y2=f/(sum(f*dy));   % pdf corresponding to each bins normalized after averaging

%----Plotting---
figure;
p=plot(y1,y2,'r-');
xlabel('money')
ylabel('frequency of money')
set(gca, 'FontSize',10,'FontWeight', 'bold','LineWidth',2 );
```

FIGURE 1.7: Code in MATLAB for generating the equilibrium distributions represented in Fig. 1.5
Source: Prepared by the authors.